

# Recursion for twisted descendants and characteristic numbers of rational curves

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## Abstract

On a space of stable maps, the psi classes are modified by subtracting certain boundary divisors. The top products of modified psi classes, usual psi classes, and classes pulled back along the evaluation maps are called *twisted descendants*; it is shown that in genus 0, they admit a complete recursion and are determined by the Gromov-Witten invariants. One motivation for this construction is that all characteristic numbers (of rational curves) can be interpreted as twisted descendants; this is explained in the second part, using pointed tangency classes. As an example, some of Schubert's numbers of twisted cubics are verified.

## Introduction

On the moduli space  $\overline{M}_{g,n}(X, \beta)$  of stable maps, the psi classes are the tautological classes  $\psi_i := c_1(\sigma_i^* \omega_\pi)$ . Here  $\pi$  denotes the projection from the tautological family and  $\sigma_i$  is the section corresponding to the  $i$ -th mark. The gravitational descendants are the top products of the psi classes and the classes pulled back along the evaluation maps. This notion was devised by E. Witten [17] in order to introduce gravity into topological sigma models. Recently there has been a lot of interest in these invariants, especially stemming from the Virasoro conjecture, which states that certain differential operators annihilate the generating function of the descendants. See Getzler [8] for the precise formulation, a survey, and a proof of the conjecture in genus 0.

In enumerative geometry, the descendants have not yet completely found their place. Since the psi classes are not invariant under pull-back via forgetful maps it is not obvious that they should have enumerative interpretation. (Note that geometrical conditions are related to the image curve, and thus should be invariant under oblivion.)

In the present note, with attention restricted to the case  $\overline{M}_{0,n}(\mathbb{P}^r, d)$ , I propose the notions of *modified psi classes*, and *twisted descendants* as vehicles for bringing the power of descendants into enumerative geometry. The exposition naturally divides into two parts:

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The first part (Sections 1–3) is on twisted descendants: The enumerative defect of a psi class  $\psi_i$  is identified as the sum  $\beta_i$  of all boundary divisors having mark  $i$  on a contracting twig; let the *modified psi class* be  $\overline{\psi}_i := \psi_i - \beta_i$ . The *twisted descendants* are then the usual gravitational descendants twisted by the modified psi classes. The main result (of the first part of the paper) is the recursive formula 3.5 for these twisted descendants, sufficient to determine them from the usual descendants, and thus from the Gromov-Witten invariants. The proof follows from a couple of results on the boundary (Lemma 2.2 and Lemma 2.3), which I also find interesting in their own right.

The second part (Section 4) is devoted to enumerative geometry. The key point is the use of pointed conditions. Let  $\Phi_i$  be the codimension-2 class of pointed tangency: for a given hyperplane  $H$ , it is the locus of maps tangent to  $H$  exactly at the mark  $i$ . It is shown (Proposition 4.1) that this locus is the zero scheme of a regular section of a vector bundle, and its class is

$$\Phi_i = \eta_i(\eta_i + \overline{\psi}_i),$$

where  $\eta_i := c_1(\nu_i^* \mathcal{O}(1))$  denotes the evaluation class. Even though these  $\Phi_i$  cycles do not in general intersect properly, it is shown (Proposition 4.7) that their top products have enumerative significance: the characteristic numbers (including compound conditions not covered by previous techniques) are all sorts of top products of  $\Phi_i$  classes (with distinct marks) and eta classes — in other words, they are twisted descendants.

As an example I compute some characteristic numbers of twisted cubics. A few of these numbers date back to Schubert and are verified for the first time; others apparently are new.

This work is a by-product of my forth-coming Ph.D. thesis; I am indebted to my advisor Israel Vainsencher for his support. Also I would like to thank Lars Ernström for introducing me to gravitational descendants, and for a very inspiring stay at Kungliga Tekniska Högskolan in Stockholm. Finally, thanks are due to Andreas Gathmann for sending me a copy of his thesis; it has been an important source of inspiration.

## 1 Preliminaries

**1.1 General conventions.** — Throughout we work over the field of complex numbers. We place ourselves in the space  $\overline{M}_{0,n}(\mathbb{P}^r, d)$  of Kontsevich stable maps in genus 0, and we always assume  $n > 0$ . For simplicity, the target space is taken to be  $\mathbb{P}^r$ ; however, all results of Sections 1–3 immediately generalise to any homogeneous variety.

If  $\mu : C \rightarrow \mathbb{P}^r$  is a stable map, the source curve  $C$  is a tree of projective lines; we will consequently refer to the irreducible components of the source curve as *twigs*, reserving the term *component* for the components of cycles in the moduli space.

Let  $\nu_i : \overline{M}_{0,n}(\mathbb{P}^r, d) \rightarrow \mathbb{P}^r$  denote the evaluation map of the  $i$ 'th mark, and let  $\eta_i := c_1(\nu_i^* \mathcal{O}(1))$  denote the pull-back of the hyperplane class.

For ulterior reference, let us collect some standard results on psi classes and gravitational descendants. For generalisations, see Getzler [7], [8], or Pandharipande [14].

**1.2 Tautological families.** — The projection  $\pi : \overline{M}_{0,n+1}(\mathbb{P}^r, d) \rightarrow \overline{M}_{0,n}(\mathbb{P}^r, d)$  that forgets the last mark, comes with  $n$  canonical sections  $\sigma_i$  defined by repeating the  $i$ 'th

mark, and stabilising. The image of  $\sigma_i$  is the boundary divisor  $\Delta_i$  having a twig of degree 0 with just the two marks  $i$  and  $n+1$ . (Throughout we will abuse of language like this when we mean: the general point of  $\Delta_i$  represents a map whose source curve has two twigs, one of which is of degree 0 and carries just the two marks  $i$  and  $n+1$ .)

The map  $\pi$ , together with the new evaluation map  $\nu_{n+1} : \overline{M}_{0,n+1}(\mathbb{P}^r, d) \rightarrow \mathbb{P}^r$ , is a tautological family over  $\overline{M}_{0,n}(\mathbb{P}^r, d)$  in the following sense: Given  $[\mu] \in \overline{M}_{0,n}(\mathbb{P}^r, d)$  representing a map  $\mu : C \rightarrow \mathbb{P}^r$ , the fibre  $\pi^{-1}([\mu])$  is a curve canonically isomorphic to  $C$ , and the sections  $\sigma_i$  single out  $n$  distinct marks on the fibre, corresponding to the distribution of marks on  $C$ . Under this identification, the restriction of  $\nu_{n+1}$  to the fibre  $\pi^{-1}([\mu])$  is exactly the map  $\mu$ .

**1.3 Psi classes.** — Let  $\omega_\pi$  be the relative dualising sheaf of  $\pi$ . The  $i$ 'th *cotangent line* of  $\overline{M}_{0,n}(\mathbb{P}^r, d)$  is the line bundle  $\sigma_i^* \omega_\pi$ . The  $i$ 'th *psi class* is its first Chern class:

$$\psi_i := c_1(\sigma_i^* \omega_\pi).$$

The psi class  $\psi_i$  can also be thought of as “minus” the normal bundle of  $\sigma_i$ :

$$\psi_i = -\sigma_i^* \Delta_i. \quad (1.3.1)$$

The psi class pulls back along the forgetful map like this:

$$\pi^* \psi_i = \psi_i - \Delta_i. \quad (1.3.2)$$

In case  $n \geq 3$  there is the following expression for the psi class:

$$\psi_1 = (1 \mid 2, 3) \quad (1.3.3)$$

where  $(1 \mid 2, 3)$  denotes the sum of boundary divisors having mark 1 on one twig and 2,3 on the other twig. This formula is a consequence of the pull-back formula above, and the obvious fact that the psi classes are trivial on  $\overline{M}_{0,3}$ .

Finally, a psi class  $\psi_i$  restricted to a boundary divisor  $\Delta$  gives the psi class  $\psi_i$  of the twig carrying  $i$ . Precisely, (see 2.1 for notation):

$$\rho_\Delta^* \psi_i = j_\Delta^* \psi_i \quad (1.3.4)$$

where it is understood that the psi class on the right hand side is pulled back from  $\overline{M}_A$  (resp.  $\overline{M}_B$ ) if  $i \in A$  (resp.  $i \in B$ ).

**1.4 Gravitational descendants.** — Top intersections of psi classes and eta classes are the *gravitational descendants*. The following notation is similar to what has become standard:

$$\langle \tau_{u_1}(c_1) \cdots \tau_{u_n}(c_n) \rangle_d := \int \psi_1^{u_1} \eta_1^{c_1} \cdots \psi_n^{u_n} \eta_n^{c_n} \cap [\overline{M}_{0,n}(\mathbb{P}^r, d)].$$

The pull-back formula 1.3.2 yields three important equations known as the puncture, dilaton and divisor equations. In each formula, the left hand side is on a space with one extra mark.

Puncture equation:

$$\langle \prod_{i=1}^n \tau_{u_i}(c_i) \cdot \tau_0(0) \rangle_d = \sum_{j=1}^n \langle \prod_{i=1}^n \tau_{u_i - \delta_{ij}}(c_i) \rangle_d$$

Dilaton equation:

$$\langle \prod_{i=1}^n \tau_{u_i}(c_i) \cdot \tau_1(0) \rangle_d = (-2 + n) \cdot \langle \prod_{i=1}^n \tau_{u_i}(c_i) \rangle_d$$

Divisor equation:

$$\langle \prod_{i=1}^n \tau_{u_i}(c_i) \cdot \tau_0(1) \rangle_d = d \cdot \langle \prod_{i=1}^n \tau_{u_i}(c_i) \rangle_d + \sum_{j=1}^n \langle \prod_{i=1}^n \tau_{u_i - \delta_{ij}}(c_i + \delta_{ij}) \rangle_d$$

In these formulae,  $\delta_{ij}$  is the Kronecker delta, and any term with a negative exponent on a psi class is defined to be 0.

**1.5 Topological recursion relation.** — Finally, the topological recursion relation applies when there are at least three marks. It is a consequence of the restriction formula 1.3.4 and the linear equivalence  $\psi_i = (i \mid j, k)$ . The formula is (assuming  $u_1 \geq 1$ ):

$$\langle \prod_{i=1}^n \tau_{u_i}(c_i) \rangle_d = \sum \langle \tau_{u_1-1}(c_1) \prod_{\substack{i \in A \\ i \neq 1}} \tau_{u_i}(c_i) \tau_0(e) \rangle_{d_A} \cdot \langle \tau_0(r-e) \prod_{i \in B} \tau_{u_i}(c_i) \rangle_{d_B}. \quad (1.5.1)$$

The sum is over all stable splittings  $A \cup B = [n]$ ,  $d_A + d_B = d$ , and over  $e = 0, \dots, r$ .

**1.6 Reconstruction for descendants.** — Together, these equations are sufficient to reduce any gravitational descendant to a Gromov-Witten invariant: Induction on the number of psi classes: if there three or more marks, use the topological recursion relation to reduce the number of psi classes. Otherwise, use first the divisor equation (backwards) to introduce more marks. (Note that this step may introduce rational coefficients.)

## 2 Boundary results

**2.1 General set-up for boundary divisors.** — To establish notation, let  $\Delta$  be the boundary divisor  $(A, d_A \mid B, d_B)$ , where  $A \cup B = [n]$  and  $d_A + d_B = d$ . Each twig corresponds to a moduli space of lower dimension,  $\overline{M}_A := \overline{M}_{A \cup x_A}(\mathbb{P}^r, d_A)$  and  $\overline{M}_B := \overline{M}_{B \cup x_B}(\mathbb{P}^r, d_B)$ , more precisely,  $\Delta$  is the image of a birational map  $\rho_\Delta : \overline{M}_A \times_{\mathbb{P}^r} \overline{M}_B \rightarrow \overline{M}_{0,n}(\mathbb{P}^r, d)$ . The fibred product is over the two evaluations maps  $\nu_{x_A} : \overline{M}_A \rightarrow \mathbb{P}^r$  and  $\nu_{x_B} : \overline{M}_B \rightarrow \mathbb{P}^r$ , reflecting the fact that in order to glue, the two maps must agree at the mark. The fibred product is a subvariety in the cartesian product  $\overline{M}_A \times \overline{M}_B$ ; let  $j_\Delta$  denote the inclusion. This set-up and notation is used throughout — summarised in the following diagram:

$$\begin{array}{ccc} & \overline{M}_{0,n}(\mathbb{P}^r, d) & \\ \uparrow \rho_\Delta & & \\ \overline{M}_A \times_{\mathbb{P}^r} \overline{M}_B & \xrightarrow{j_\Delta} & \overline{M}_A \times \overline{M}_B. \end{array}$$

**2.2 Lemma.** — *Let  $\Delta = (A, d_A \mid B, d_B)$  be a boundary divisor such that  $A$  is non-empty and  $d_A < d_B$ . Then  $\rho_\Delta$  is an isomorphism onto  $\Delta$ .*

This lemma completes Lemma 12 of FP-NOTES [5], where it is proved that if both  $A$  and  $B$  are non-empty then  $\rho_\Delta$  is an isomorphism onto its image. Together the two lemmas cover all the cases of isomorphism.

Note in particular that whenever a boundary divisor  $\Delta$  has a twig of degree zero, the map  $\rho_\Delta$  is an isomorphism onto  $\Delta$ .

*Proof.* — We need to show that given a map  $\mu : C \rightarrow \mathbb{P}^r$  in the divisor  $\Delta$ , there is only one node (referred to as the official node) that can be the limit of the node of generic elements in  $\Delta$ ; in other words, there is only one way to cut the curve.

START: Due to the lemma of FP-NOTES, we can suppose  $B = \emptyset$ . There is a minimal sub-tree  $T_A \subset C$  carrying all the marks. Let  $T_A$  grow:

STEP 1: Whenever  $T_A$  disconnects the rest, then there is only one of the remaining connected component that has total degree  $\geq d_B$ ; let  $T_A$  grow, incorporating the others.

STEP 2: As long as the total degree of  $T_A$  is less than  $d_A$ , we must incorporate the next twig. If this act disconnects the rest, repeat step 1.

STOP: If the total degree of  $T_A$  is equal to  $d_A$  the growth stops here, and we have found the unique spot to cut the curve. Indeed, if the next twig  $C'$  has positive degree, it is clear that we cannot include it in  $T_A$ . If  $C'$  has degree zero, then by *stability* it must intersect another two connected components, each of which must have positive total degree. Including  $C'$  would disconnect the rest, and consequently force one of the remaining components to be included in  $T_A$ . Anyway, the degree of  $T_A$  would become too high.

This shows that the map  $\rho_\Delta$  is bijective onto its image. The proof of normality of  $\Delta$  is identical to that indicated in FP-NOTES and resorts to the locally rigidified moduli space  $\overline{M}_{0,n}(\mathbb{P}^r, d, \bar{t})$ : For the divisors lying over  $\Delta$  to intersect, there must be a symmetry along some degeneration of the configuration represented by  $\Delta$ . The above arguments show such symmetries do not occur, so  $\Delta$  is normal. Since  $\rho_\Delta$  is bijective between normal varieties (and we are in characteristic zero) it must be an isomorphism.  $\square$

**2.3 Lemma.** — *If  $\Delta$  is a boundary divisor such that  $\rho_\Delta$  is an isomorphism onto its image, then*

$$\rho_\Delta^* \Delta = j_\Delta^* (-\psi_{x_A} - \psi_{x_B}).$$

Here  $\psi_{x_A}$  (resp.  $\psi_{x_B}$ ) denotes the pull-back to  $\overline{M}_A \times \overline{M}_B$  from  $\overline{M}_A$  (resp.  $\overline{M}_B$ ) of the psi class of the gluing point.

This generalises 1.3.1. Note in particular that by Lemma 2.2, the formula holds for all boundary divisors with a contracting twig.

For the sake of completeness: If  $\Delta$  is not normal, its pull-back along  $\rho_\Delta$  has one more term which is the push-forth of the singular locus of  $\Delta$ . I will not prove that formula here, since it is not needed in the sequel.

*Proof.* — For short, set  $\overline{M} := \overline{M}_{0,n}(\mathbb{P}^r, d)$  and  $U := \overline{M}_{0,n+1}(\mathbb{P}^r, d)$ , and let  $\pi : U \rightarrow \overline{M}$  be the forgetful map. Its restriction  $\bar{\pi}$  to  $D_\Delta := \pi^{-1}(\Delta)$  is clearly a tautological family over  $\Delta$ ; note that  $D_\Delta$  is a divisor in  $U$  and it has two components  $D_A$  and  $D_B$ , corresponding

to whether the new mark is on the  $A$ -twig or the  $B$ -twig. Now the key point is that the family  $\bar{\pi} : D_\Delta \rightarrow \Delta$  has a section  $\bar{\sigma}_x : \Delta \rightarrow D_\Delta$ , defined by putting the new mark on the official node and stabilising. This works because  $\Delta$  has only one official node, according to (the proof of) the previous lemma.

$$\begin{array}{ccc} D_\Delta & \xhookrightarrow{\bar{\rho}} & U \\ \bar{\sigma}_x \uparrow \downarrow \bar{\pi} & & \downarrow \pi \\ \Delta & \xhookrightarrow{\rho} & \overline{M} \end{array}$$

Note that  $D_A$  and  $D_B$  intersect transversely along  $\bar{\sigma}_{x*}\Delta$ .

From the cartesian diagram we get

$$\begin{aligned} \rho^* \rho_* \Delta &= \rho^* (\pi_* \bar{\rho}_* \sigma_{x*} \Delta) \\ &= \bar{\pi}_* \bar{\rho}^* (\bar{\rho}_* \sigma_{x*} \Delta) \\ &= \bar{\pi}_* \bar{\rho}^* (D_A \cap D_B) \\ &= \bar{\pi}_{A*} \bar{\rho}_A^* (D_A \cap D_B) + \bar{\pi}_{B*} \bar{\rho}_B^* (D_A \cap D_B), \end{aligned} \tag{2.3.1}$$

where the subscripted maps in the last line are explained by the diagram

$$\begin{array}{ccccc} & & U & & \\ & \nearrow \bar{\rho}_A & \uparrow \bar{\rho} & \nwarrow \bar{\rho}_B & \\ D_A & \xhookrightarrow{\quad} & D_\Delta & \xhookleftarrow{\quad} & D_B \\ & \searrow \bar{\pi}_A & \downarrow \bar{\pi} & \swarrow \bar{\pi}_B & \\ & & \Delta & & \end{array}$$

Note that the section  $\bar{\sigma}_x$  factors through  $D_A$  giving a section  $\bar{\sigma}_{x_A} : \Delta \rightarrow D_A$ . The image of  $\bar{\sigma}_{x_A}$  (which we denote by the same symbol) is cut out by the divisor  $D_B$  restricted to  $D_A$ .

So, concerning the first term in 2.3.1, we continue

$$\begin{aligned} \bar{\pi}_{A*} (\bar{\rho}_A^* (D_A \cap D_B)) &= \bar{\pi}_{A*} (\bar{\rho}_A^* D_A \cap \bar{\sigma}_{x_A}) \\ &= \bar{\sigma}_{x_A}^* (\bar{\rho}_A^* D_A) \\ &= \bar{\sigma}_{x_B}^* (\bar{\rho}_B^* D_A) \\ &= \bar{\sigma}_{x_B}^* (\bar{\sigma}_{x_B}). \end{aligned}$$

By assumption we are identifying  $\Delta$  with the fibred product  $\overline{M}_A \times_{\mathbb{P}^r} \overline{M}_B$ ; under this identification,  $\bar{\pi}_A : D_A \rightarrow \Delta$  is the pull-back along  $j_\Delta$  of the tautological family  $\pi_A : U_A \rightarrow \overline{M}_A$ . Similarly for  $B$ . The intersection  $D_A \cap D_B$  in  $U$  corresponds to the image of the sections  $\sigma_{x_A}$  and  $\sigma_{x_B}$ , along which the pulled-back families  $j_\Delta^* U_A$  and  $j_\Delta^* U_B$

are glued together. (Since the tautological families do not have the universal property, these identifications are only natural up to non-unique isomorphism, but that is sufficient for our purposes.)

We find that

$$\begin{aligned}\bar{\sigma}_{x_B}^*(\bar{\sigma}_{x_B}) &= j_{\Delta}^* \sigma_{x_B}^* \sigma_{x_B} \\ &= -j_{\Delta}^* \psi_{x_B}.\end{aligned}$$

Same reasoning for the second term in 2.3.1 yields the result.  $\square$

*Definition.* — On a space  $\overline{M}_{0,n}(\mathbb{P}^r, d)$  with  $d > 0$ , let  $\beta_i$  denote the union of all boundary divisors whose general point represents a map contracting the twig carrying the mark  $i$ . That is,  $\beta_i = \sum_{A \ni i} (A, 0 \mid B, d)$ .

**2.4 Lemma.** — Let  $\Delta = (A, 0 \mid B, d)$  be a boundary divisor with a contracting twig.

(a) If  $i \in A$  then

$$\rho_{\Delta}^* \beta_i = j_{\Delta}^* ((i \mid x_A) - \psi_{x_A} - \psi_{x_B} + \beta_{x_B}).$$

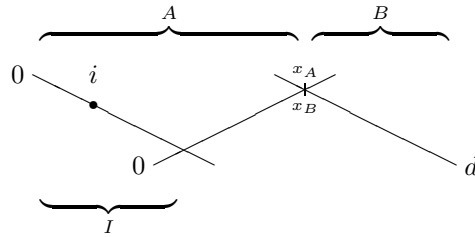
(b) If  $i \in B$  then

$$\rho_{\Delta}^* \beta_i = j_{\Delta}^* \beta_i.$$

In (a), the symbol  $(i \mid x_A)$  denotes the union of all boundary divisors of  $\overline{M}_A$  with mark  $i$  on one twig and  $x_A$  on the other. The divisor  $\beta_{x_B}$  lives in  $\overline{M}_B$ . In (b), the divisor  $\beta_i$  is understood to live in  $\overline{M}_B$ , since  $i \in B$ .

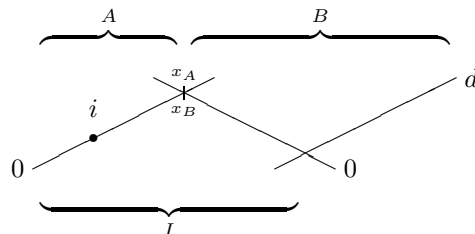
*Proof.* — Ad. (a): Note first of all that  $\Delta$  itself is a component in  $\beta_i$ . Therefore the product includes the self-intersection class  $\Delta^2 = j^*(-\psi_{x_A} - \psi_{x_B})$  (by 2.3). The other components of  $\beta_i$  are boundary divisors distinct from  $\Delta$ , and their intersection with  $\Delta$  is a reduced effective cycle. For such a divisor, let  $I \subset [n]$  denote the degree 0 part, that is, the part containing  $i$ . If neither  $I \subsetneq A$  nor  $I \supsetneq A$  then clearly the intersection is empty.

Each component of  $\beta_i$  such that  $I \subsetneq A$ , intersects  $\Delta$  by breaking the  $A$ -twig like this:



such that the mark  $i$  is broken apart from the attachment mark  $x_A$ . There is a 1–1 correspondence between the possible  $I \subsetneq A$  and these possible breaks of the  $A$ -twig. These codimension-2 boundary cycles correspond again to divisors in  $\overline{M}_A$  such that  $i$  is on one twig and  $x_A$  is on the other twig. This accounts for the contribution  $(i \mid x_A)$ .

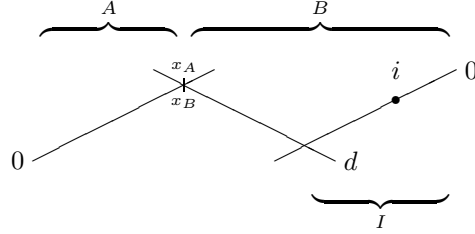
Next, the cases  $I \supsetneq A$  corresponds to breaking the  $B$ -twig in this way:



such that the marks of  $I \cap B$  are put on the new middle twig, of degree 0. In the space  $\overline{M}_B$  this corresponds to all ways of breaking off a twig of degree 0 carrying the mark  $x_B$ . So these cases give the contribution  $\beta_{x_B}$ .

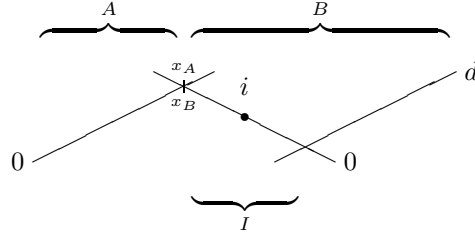
Now for item (b): This time,  $\Delta$  has  $i$  on the degree  $d$  twig, so all components in  $\beta_i$  are distinct from  $\Delta$ . The intersection of such a divisor with  $\Delta$  is empty unless  $I \subset B$  or  $I \supset A \cup \{i\}$ .

The cases  $I \subset B$  corresponds to breaking the  $B$ -twig like this



— that is: break off a degree 0 twig containing  $i$  but not  $x_B$ .

The cases  $I \supset A \cup \{i\}$  corresponds to breaking off a degree 0 twig containing both  $i$  and  $x_B$ .



All together we get all the ways of breaking off  $i$  with degree 0, that is, the contribution is  $\beta_i$  (from the  $B$ -space).  $\square$

### 3 Modified psi classes

*Definition.* — On a space  $\overline{M}_{0,n}(\mathbb{P}^r, d)$  with  $d > 0$ , the *modified psi classes* are by definition

$$\overline{\psi}_i := \psi_i - \beta_i.$$

The motivation for this definition is the following observation:

**3.1 Remark.** — Let  $\pi : \overline{M}_{0,n+1}(\mathbb{P}^r, d) \rightarrow \overline{M}_{0,n}(\mathbb{P}^r, d)$  be the universal map that forgets the last mark. Then

$$\pi^* \overline{\psi}_i = \overline{\psi}_i.$$

**3.2 Remark.** — On a space with just one mark, we have  $\overline{\psi}_1 = \psi_1$ , so an equivalent definition is  $\overline{\psi}_i := \hat{\pi}_i^* \psi_i$ , where  $\hat{\pi}_i$  denotes the map that forgets all marks except  $i$ .

**3.3 Key formula.** — Let  $\Delta = (A, 0 \mid B, d)$  be a boundary divisor with a contracting twig. Then

$$\rho_\Delta^* \overline{\psi}_i = \begin{cases} j_\Delta^* \overline{\psi}_{x_B} & \text{in case } i \in A \\ j_\Delta^* \overline{\psi}_i & \text{in case } i \in B. \end{cases}$$



*Proof.* — By definition,  $\overline{\psi}_i = \psi_i - \beta_i$ . Suppose  $i \in A$ . Applying the restriction formula 1.3.4 and the previous lemma gives  $\sigma_\Delta^* \overline{\psi}_i = j_\Delta^* (\psi_i - (i \mid x_A) + \psi_{x_A} + \psi_{x_B} - \beta_{x_B})$ . Now note that by stability, the  $A$  twig must have yet another mark, say  $j$ , so by 1.3.3 we can write  $\psi_i = (i \mid j, x_A)$ , and  $\psi_{x_A} = (i, j \mid x_A)$ . It follows that  $\psi_i + \psi_{x_A} = (i \mid x_A)$ . This proves the first case. The second case is immediate from the lemma.  $\square$

The second case shows the geometric nature of the modified classes: If two marks come together, a new degree zero twig sprouts out and takes the marks, but the corresponding modified psi classes stick to the original twig instead of following the marks to the new one, which is illusory from the point of view of the image curve.

## Recursion of twisted descendants

*Definition.* — We define a *twisted descendant* to be a top product of modified psi classes, usual psi classes, and the eta classes, and agree on the following notation:

$$\langle \tau_{u_1}^{m_1}(c_1) \cdots \tau_{u_n}^{m_n}(c_n) \rangle_d := \int \psi_1^{u_1} \overline{\psi}_1^{m_1} \eta_1^{c_1} \cdots \psi_n^{u_n} \overline{\psi}_n^{m_n} \eta_n^{c_n} \cap [\overline{M}_{0,n}(\mathbb{P}^r, d)].$$

**3.4 Remark.** — Since the modified psi classes are invariant under pull-back along forgetful maps, the puncture, dilaton and divisor equations are also valid for twisted descendants: the modified psi classes go through untouched. The three equations are not needed in the recursion below.

**3.5 Recursion relation.** — Suppose  $m_1$ , the exponent of the first modified psi class, is positive. Then

$$\langle \prod_{i=1}^n \tau_{u_i}^{m_i}(c_i) \rangle_d = \langle \tau_{u_1+1}^{m_1-1}(c_1) \prod_{i=2}^n \tau_{u_i}^{m_i}(c_i) \rangle_d - \sum_{i \in A} \langle \prod_{i \in A} \tau_{u_i} \cdot \tau_0 \rangle \cdot \langle \tau_0^{\mathbf{m}-1}(\mathbf{c}) \cdot \prod_{i \in B} \tau_{u_i}^{m_i}(c_i) \rangle_d$$

The sum is over all partitions  $A \cup B = [n]$  with  $1 \in A$  and  $\sharp A \geq 2$ . In the last factor, we have used the short hand notation  $\mathbf{m} := \sum_{i \in A} m_i$ , the sum of all the exponents of the modified psi classes corresponding to marks in  $A$ , and similarly we have set  $\mathbf{c} := \sum_{i \in A} c_i$ .

The product  $\langle \prod_{i \in A} \tau_{u_i} \cdot \tau_0 \rangle$  is on the Knudsen-Mumford space  $\overline{M}_{0,A \cup x_A}$  of stable pointed curves with labels  $A \cup x_A$ . The top products of psi classes on these spaces are well known: if  $n_A := \sharp A$  then the factor evaluates to

$$\frac{(n_A - 2)!}{\prod_{i \in A} u_i!}$$

*Proof.* — The left hand side has a factor  $\overline{\psi}_1 = \psi_1 - \beta_1$ , so we can write it as

$$\psi_1 \cdot \text{rest} - \sum \rho_\Delta^* \text{rest}$$

where the sum is over all components  $\Delta$  of  $\beta_1$ . Applying the formulae for  $\rho_\Delta$ , the sum is

$$\sum \int_\Delta j_\Delta^* \left( \overline{\psi}_{x_B}^{\mathbf{m}-1} \prod_{i \in B} \overline{\psi}_i^{m_i} \prod_{i=1}^n \psi_i^{u_i} \eta_i^{c_i} \right).$$

Pulling back via  $j_\Delta$  amounts to multiplying with the Künneth decomposition of the diagonal (pulled back via the evaluation maps of the gluing marks  $x_A$  and  $x_B$ ), and then we can distribute the terms, according to which space they are pulled back from. We get

$$\sum \int_{\overline{M}_A} \left( \prod_{i \in A} \psi_i^{u_i} \eta_i^{c_i} \cdot \eta_{x_A}^e \right) \cdot \int_{\overline{M}_B} \left( \overline{\psi}_{x_B}^{m-1} \eta_{x_B}^{r-e} \cdot \prod_{i \in B} \overline{\psi}_i^{m_i} \psi_i^{u_i} \eta_i^{c_i} \right)$$

where the sum is also over  $e = 0, \dots, r$ .

Finally note that  $\overline{M}_A = \overline{M}_{0, A \cup x_A} \times \mathbb{P}^r$ , so the only non-zero contributions come when  $e = r - \mathbf{c}$ .  $\square$

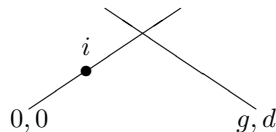
**3.6 Corollary.** — *The twisted descendants can be reconstructed from the usual gravitational descendants (and thus from the Gromov-Witten invariants).*

*Proof.* — The recursive formula above expresses each twisted descendant as a sum of twisted descendants with either fewer modified psi classes or fewer marks. When we come down to 1-pointed space, the modified psi classes are equal to the usual psi classes, and then we have a usual gravitational descendant. (From gravitational descendants to Gromov-Witten invariants we have 1.6.)  $\square$

**3.7 Remark.** — Kontsevich and Manin [12] also studied certain twisted descendants (called generalised correlators) and obtained a recursion ([12], Theorem 1.2) visually similar to mine. Instead of twisting with  $\overline{\psi}_i$  they twisted with  $\phi_i := \text{st}^* \psi_i$ , where  $\text{st}$  is the map to Knudsen-Mumford space consisting in forgetting the map and stabilising. Note that  $\phi_i$  is also a modification of  $\psi_i$ : the difference consists in all boundary divisors such that  $i$  is alone on a twig.

It is interesting to compare the two recursions: The recursion of the present note consists in breaking off twigs containing  $i$  and of *degree zero*, resulting in a sum over all possible distributions of the remaining marks. The coefficients are integrals over Knudsen-Mumford spaces. The recursion of Kontsevich and Manin consists in breaking off twigs containing *only* the mark  $i$ ; the resulting sum is over all possible degrees of this twig, and the coefficients are two-pointed descendants.

**3.8 Possible generalisations.** — Let me make an informal comment on possible generalisations. As already noted, all results hitherto remain valid when  $\mathbb{P}^r$  is replaced by any homogeneous variety  $X$ . Even though the boundary analysis of Section 2 relies on the specific geometry of  $\overline{M}_{0,n}(X, \beta)$  in this case, I believe the results hold also in higher genus and for general smooth projective  $X$ . The correction  $\beta_i$  should then be the divisor (in the appropriate component of  $\overline{M}_{g,n}(X, \beta)$ ) defined as the sum of boundary divisors of type



The correspondingly defined modified psi class should then satisfy the Key Formula 3.3, and the Recursion Relation 3.5 would follow (with virtual fundamental class instead of the topological one, and maybe with signs arising from odd cohomology... ). Note that the coefficients  $\langle \prod \tau_{u_i} \cdot \tau_0 \rangle$  would still live on the genus 0 space  $\overline{M}_{0,A \cup x_A}$  even in the higher genus case. Such a generalisation would be particularly useful in genus 1 and 2 where topological recursion is available (cf. Getzler [7]), in which case also the Corollary 3.6 would hold.

## 4 Tangency conditions and characteristic numbers

The motivation for studying twisted descendants comes from enumerative geometry: This final section shows that characteristic numbers of rational curves in  $\mathbb{P}^r$  are twisted descendants. The characteristic numbers are those one can make out of

- incidence conditions, and
- conditions of tangency to a hyperplane  $H \subset \mathbb{P}^r$  at a specified linear subspace of  $H$ .

The central idea is the use of pointed conditions.

Previously, working without marks, Pandharipande [13] gave an algorithm for computing top intersection products of incidence divisors and tangency divisors, allowing for the computation of the simple characteristic numbers. This method comes short in treating compound conditions, since without marks there is not enough control over the tangencies.

The case of  $\mathbb{P}^2$  has always been object of particular interest. Let  $N_{a,b}$  denote the number of degree  $d$  rational plane curves incident to  $a$  points and tangent to  $b$  lines ( $a + b = 3d - 1$ ). Pandharipande has shown that the corresponding generating function

$$G(x, y, z) = \sum \sum N_{a,b} y^a x^b \exp(dz)/a!b!$$

satisfies the partial differential equation

$$G_{yz} = -G_x + G_{xz} - (1/2)G_{zz}^2 + (G_{zz} + yG_{xz})^2.$$

This is the content of the famous unpublished e-mail to Lars Ernström [15]. The proof is forthcoming in Graber & Pandharipande [9].

In another line of development, Ernström and Kennedy [2], constructed *the space of stable lifts* of maps to  $\mathbb{P}^2$ , and computed characteristic numbers of rational plane curves, involving also the codimension-2 condition of being tangent to a given line at a specified point. (It is not clear whether the same techniques apply also in higher dimensions.)

In a later paper [3] they showed how this data is encoded in a ring called the *contact cohomology ring*, which specialises to the usual cohomology ring of  $\mathbb{P}^2$  when certain formal parameters are set to zero.

They also show how the recursion can be expressed in terms of partial differential equations. It would be very interesting to see if there is a relation between my recursion of twisted descendants, and these partial differential equations.

## Pointed tangency condition

In this section we assume  $d \geq 2$  and  $r \geq 2$ . (The propositions 4.1, 4.3 and 4.7 hold also in the case  $d = 1$ , but require slightly different proofs.)

Let  $H \subset \mathbb{P}^r$  be a hyperplane, given as the zero scheme of a linear form  $f \in H^0(\mathbb{P}^r, \mathcal{O}(1))$ . We consider first the case where there is only one mark. The mark of a map  $\mu : C \rightarrow \mathbb{P}^r$  lands in  $H$  if and only if  $\mu^*f$  vanishes at the mark.

*Definition.* — A map  $\mu : C \rightarrow \mathbb{P}^r$  with a single mark is *tangent* to  $H$  at the mark if and only if  $\mu^*f$  vanishes at the mark with multiplicity 2, i.e., when also the first derivative vanishes. Let  $\Phi_1(H)$  denote the locus of maps tangent to  $H$  at the mark. (Shortly we will see that the reduced scheme structure is the natural one.)

In the presence of more marks, we must make the following reservation: in this case, the boundary divisor  $\beta_1$  is non-empty, so there exist maps whose source curve is reducible with the first mark on a degree 0 twig, and such that the mark (and thus the twig) maps to a point in  $H$ . For such a map,  $\mu^*f$  vanishes identically along the contracting twig and thus to any order. But we do not in general want to consider such maps tangent.

Note that these maps have their source collapsed under the forgetful map  $\hat{\pi}_1$  that forgets all marks but the first, so a technically convenient way to characterise pointed tangency in the case of more marks is this:

*Definition.* — A map  $\mu : C \rightarrow \mathbb{P}^r$  is tangent to  $H$  at the first mark if its image under  $\hat{\pi}_1$  is tangent to  $H$  at the mark. Let  $\Phi_1(H)$  be the locus of such maps. Accordingly,

$$\Phi_1(H) = \hat{\pi}_1^* \Phi_1(H), \quad (4.0.1)$$

where the  $\Phi_1$  cycle on the right hand side lives on the space with only one mark (compare 3.2).

**4.1 Proposition.** — *The locus  $\Phi_i(H)$  is the zero scheme of a regular section of a rank two vector bundle  $E$ , and its class is*

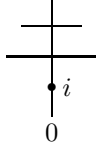
$$\Phi_i := c_2(E) = \eta_i(\eta_i + \overline{\psi}_i).$$

Since we are going to intersect various  $\Phi_i$  cycles, we will need a slightly more general set-up.

*Definition.* — A one-marked map  $\mu : C \rightarrow \mathbb{P}^r$  has *critical mark* if its differential vanishes at the mark. The locus of such maps is denoted  $\Gamma_1$ . As in the definition of  $\Phi_1$ , we don't want to consider the maps in  $\beta_1$  as having critical mark, so in a situation with more marks, we define a map to have to have critical first mark if its image under  $\hat{\pi}_1$  has critical mark. It follows that

$$\Gamma_1 = \hat{\pi}_1^* \Gamma_1.$$

Note that any map with source of type



has critical  $i$ 'th mark. (Here the 0 indicates the degree of the vertical twig.)

**4.2 Families of maps.** — Given a flat family of stable maps

$$\begin{array}{ccc}
 \mathfrak{X} & \xrightarrow{\mu} & \mathbb{P}^r \\
 \sigma_1 \uparrow & \downarrow \pi & \nearrow \nu_1 \\
 S, & & 
 \end{array}$$

consider the following conditions:

- (A)  $S$  is equidimensional and reduced.
- (B) The locus of maps with reducible source is of pure codimension 1 in  $S$  (or it is empty).
- (C) For each mark  $i$ , the locus  $\Gamma_i \subset S$  has codimension at least 2.

Clearly,  $\overline{M}_{0,n}(\mathbb{P}^r, d)$  satisfies all three conditions.

**4.3 Proposition.** — *In any family satisfying (A), (B) and (C), for general  $H$ , the scheme  $\Phi_1(H)$  is of codimension 2 and satisfies (A) and (B) (or it may be empty).*

*Proof.* — Let  $\hat{\pi}_1$  denote the map that forgets all marks but the first. Since  $\Phi_1(H) = \hat{\pi}_1^* \Phi_1(H)$ , and since the properties are preserved under flat pull-back, it is enough to prove the proposition in the case of only one mark.

Consider the pull-back of  $f$  along  $\mu$ , and its first jet relative to  $\pi$ , (i.e. we derive only in vertical direction). The vanishing of this section

$$\partial_\pi^1 \mu^* f \in J_\pi^1 \mu^* \mathcal{O}(1)$$

defines the locus of maps tangent to  $H$ . Pulling back along  $\sigma_1$  we get only those maps tangent to  $H$  at the mark, so set-theoretically  $\Phi_1(H)$  is the zero locus of the section

$$\varphi := \sigma_1^* \partial_\pi^1 (\mu^* f)$$

of the rank-2 vector bundle

$$E := \sigma_1^* J_\pi^1 (\mu^* \mathcal{O}(1)).$$

(Note that  $\pi$  is not a smooth map, so a priori the sheaf  $J_\pi^1 (\mu^* \mathcal{O}(1))$  needs not be locally free. But  $\pi$  is smooth in a neighbourhood of the image of  $\sigma_1$ ; therefore  $E$  is indeed a vector bundle.)

Off  $\Gamma_1$ , the vector bundle  $E$  is generated by global sections (namely the value and the value of the derivative), so by Kleiman's Bertini theorem (Remark 5 and 6 in [10]), for

a general  $H$ , the zero-scheme  $Z(\varphi) \setminus \Gamma_1$  is reduced of pure codimension 2 (or else it is empty).

Now  $\Phi_1(H)$  is contained in the divisor  $\eta_1(H)$  of maps whose mark lands in  $H$ ; this divisor moves in a linear system without base points, so the intersection  $\Gamma_1 \cap \eta_1(H)$ , and hence the intersection  $\Gamma_1 \cap Z(\varphi)$ , is of codimension (at least) 3 in  $S$ . It follows that the whole of  $Z(\varphi)$  is in fact reduced of pure codimension 2, so  $\Phi_1(H) = Z(\varphi)$  satisfies (A).

To see that  $\Phi_1(H)$  satisfies (B), let  $B \subset S$  denote the locus of maps with reduced source, and apply the Kleiman-Bertini to  $B \setminus \Gamma_1$ . It follows that for general  $H$ , the intersection  $\Phi_1(H) \cap B \subset S$  is of codimension 3, away from  $\Gamma_1$ . But anyway  $\Phi_1(H) \cap \Gamma_1 \subset S$  is of codimension 3, so the dimensionality holds everywhere.  $\square$

**4.4 Remark.** — The last dimension reduction argument of the proof above holds for any subvariety  $D \subset S$  of codimension at least 1: For general  $H$ , the intersection  $\Phi_1(H) \cap D$  is of codimension at least 3 in  $S$ .

**4.5 Remark.** — Note that  $\overline{M}_{0,n}(\mathbb{P}^r, d)$  is Cohen-Macaulay. Being locally the quotient of a smooth variety by a finite group, its Cohen-Macaulay-ness follows from the following algebraic result, valid in characteristic zero: — *If  $R$  is a Cohen-Macaulay ring, acted upon by a finite group  $G$ , then the ring of invariants  $R^G$  is also Cohen-Macaulay.* (See Bruns and Herzog [1], Cor. 6.4.6.)

*Proof of Proposition 4.1.* — Suppose first there is only one mark, and continue the notation of the proof above, with  $S = \overline{M}_{0,1}(\mathbb{P}^r, d)$ . Since  $Z(\varphi)$  is of expected codimension and  $\overline{M}_{0,1}(\mathbb{P}^r, d)$  is Cohen-Macaulay, the section  $\varphi \in H^0(S, E)$  is regular, and  $[Z(\varphi)] = c_2(E) \cap [\overline{M}_{0,1}(\mathbb{P}^r, d)]$ . This top Chern class is easily computed via the fundamental jet bundle exact sequence (pulled back along  $\sigma_1$ )

$$0 \longrightarrow \sigma_1^*(\mu^*\mathcal{O}(1) \otimes \omega_\pi) \longrightarrow \sigma_1^*(J_\pi^1 \mu^*\mathcal{O}(1)) \longrightarrow \sigma_1^*\mu^*\mathcal{O}(1) \longrightarrow 0;$$

the last term is just  $\nu_1^*\mathcal{O}(1)$ , with first Chern class  $\eta_1$ , and the first term is also a line bundle, with first Chern class  $\eta_1 + \psi_1$ . Since  $\psi_1 = \overline{\psi}_1$  on a space with just one mark, this proves the proposition in case  $n = 1$ .

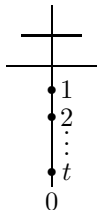
In the case of more marks, take the vector bundle to be the pull-back of  $E$  along the forgetful map  $\hat{\pi}_1$ . Since the classes  $\Phi_1$ ,  $\overline{\psi}_1$  and  $\eta_1$  are preserved under this pull-back, the result follows.  $\square$

**4.6 Remark.** — Invoking the Cohen-Macaulay-ness of  $\overline{M}_{0,n}(\mathbb{P}^r, d)$  is not strictly necessary to obtain the class of  $\Phi_1$ ; in fact the formula  $\Phi_1 = \eta_1(\eta_1 + \overline{\psi}_1)$  holds in any family satisfying (A), (B) and (C). To see this, recall (from Fulton [4], 14.1) that there is a localised top Chern class  $\mathbf{Z}(\varphi) \in A_*(Z(\varphi))$  whose image in  $A_*(S)$  is  $c_2(E) \cap [S]$ ; when  $Z(\varphi)$  has expected codimension, with cycle  $[Z(\varphi)] = \sum m_i [D_i]$ , then  $\mathbf{Z}(\varphi) = \sum e_i [D_i]$ , and  $1 \leq e_i \leq m_i$  ([4], Example 14.4.4). In our case,  $Z(\varphi)$  is of correct dimension and is furthermore *reduced*, hence  $[Z(\varphi)] = \mathbf{Z}(\varphi) = c_2(E) \cap [S]$ .

## Characteristic numbers

We will show that the top intersections of Phi classes (with distinct marks) together with eta classes have enumerative significance. The key result is that for general choices of hyperplanes  $H$ , the corresponding top scheme theoretic intersection is reduced and (at most) zero dimensional.

However, the transversality argument has an interesting extra twist due to the phenomenon of *illusory excess*: For any  $3 \leq t \leq r$ , the intersection cycle  $\Phi_1 \cdots \Phi_t$  (which ought to be of codimension  $2t$ ) has excess along the locus of maps with source



where the vertical twig has degree 0 and maps to the intersection of the  $t$  hyperplanes. This locus has codimension only  $t + 2$ . However this is illusory excess in the sense that the excessive freedom is just the freedom of marks moving on a contracting twig. The corresponding image curves move much less. Indeed, we can forget  $t - 1$  marks without destabilising, so in the space with no marks, the bad locus gets codimension  $t + 1$ , while the image of the cycle  $\Phi_1 \cdots \Phi_t$  has codimension  $t$ . If the total intersection is of top degree, a dimension reduction argument then shows that the bad curves do not occur at all! The proof of the Lemma below shows how to cope with the phenomenon.

**4.7 Proposition.** *Enumerative significance.* — Let  $Z \subset \overline{M}_{0,n}(\mathbb{P}^r, d)$  denote the scheme theoretic intersection of Phi loci and eta divisors, such that for each mark there is at most one Phi, and such that the total codimension of the cycles equals  $\dim \overline{M}_{0,n}(\mathbb{P}^r, d)$ . Then for general hyperplanes  $H$ , the intersection  $Z$  is reduced of dimension 0 (or it may be empty).

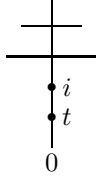
Furthermore, the corresponding finitely many maps are immersions with irreducible source (so in particular they have no automorphisms), and they are only simply tangent to the given hyperplanes.

The proof will follow easily from the following result:

**4.8 Lemma.** — In a  $2t$ -parameter family satisfying (A), (B) and (C), for general hyperplanes  $H_i$ , the scheme-theoretic intersection  $\Phi_1(H_1) \cap \cdots \cap \Phi_t(H_t)$  is reduced of dimension 0 (or it may be empty).

*Proof of the Lemma.* — Induction on  $t$ . If  $t = 1$  then the result follows from Proposition 4.3. Suppose the lemma holds for families of dimension  $2t - 2$  and consider a family  $S$  of dimension  $2t$ . For short, set  $\Phi_i = \Phi_i(H_i)$ .

Again by the Proposition 4.3, the variety  $\Phi_t$ , with the induced family, has dimension  $2t - 2$  and satisfies (A) and (B). However, it will not in general satisfy (C), because the locus  $P$  of maps with source



and with the vertical contracting twig mapping to  $H_t$ , has codimension 3 in  $S$ . These maps are tangent to  $H_t$  and have critical  $i$ 'th mark.

But we can circumvent this obstacle by forgetting the  $t$ 'th mark: Since  $P$  has at least four special points on the contracting twig, its image under  $\pi_t$  retains the codimension 3 with respect to the whole space, and therefore it has codimension 2 with respect to  $\pi_t(\Phi_t)$ .

$$\begin{array}{ccccc}
 P & \overset{\text{codim } 1}{\subset} & \Phi_t & \overset{\text{codim } 2}{\subset} & S \\
 \downarrow & & \downarrow & & \\
 \pi_t(P) & \overset{\text{codim } 2}{\subset} & \pi_t(\Phi_t) & & 
 \end{array}$$

So  $\pi_t(\Phi_t)$  satisfies (C), and since  $\Phi_t \rightarrow \pi_t(\Phi_t)$  is generically one-to-one, the image also satisfies (A) and (B).

By induction, for general choices of hyperplanes  $H_1, \dots, H_{t-1}$ , the intersection  $\Phi_1 \cap \dots \cap \Phi_{t-1}$  in the space  $\pi_t(\Phi_t)$  is reduced of dimension zero, and by the dimension reduction argument 4.4 all these finitely many points correspond to simple tangencies. Now in an open subset around these points, the map  $\Phi_t \rightarrow \pi_t(\Phi_t)$  is immersive and one-to-one, so we conclude that also the intersection upstairs  $\Phi_1 \cap \dots \cap \Phi_{t-1} \cap \Phi_t$  in the space  $S$  is reduced.  $\square$

*Proof of Proposition 4.7.* — The space  $\overline{M}_{0,n}(\mathbb{P}^r, d)$  satisfies the assumptions (A), (B) and (C). The intersection of all the eta divisors will also satisfy the three conditions, since the divisors move in base point free linear systems. We have arrived at a  $2t$ -parameter family with  $t$  Phi conditions, so the lemma applies.

As to the further properties, they follow from the dimension reduction argument: In each step, the loci of maps with reducible source, critical marks, or non-simple tangency (flexes or bitangents) have codimension at least 1, so when we come down to dimension zero they no longer occur.  $\square$

Since the intersection is reduced of dimension zero, the number of points in it is equal to the top product of the corresponding classes. The enumerative meaning is this. Each mark  $i$  corresponds to a condition; the factor  $\Phi_i \eta_i^{c_i}$  corresponds to the condition of being tangent to a given hyperplane  $H_i$  at a specified linear subspace of codimension  $c_i$  in  $H_i$ . This understood, the Proposition implies that

**4.9 Corollary.** — *The top products of Phi classes (with distinct marks) and eta classes count what they are expected to.*

By Proposition 4.1, these numbers are top products of modified psi classes and eta classes, so they are twisted descendants, and can be computed by the recursion described in 3.6. I have implemented the algorithm in Maple — the code is available upon request.



**4.10 Remark.** (Relation with the unmarked conditions.) — Let  $\iota$  be the incidence divisor (of maps incident to a given codimension-2 plane), and let  $\tau$  be the tangency divisor (of maps tangent to a given hyperplane), as defined by Pandharipande [13].

Let  $\pi_i : \overline{M}_{0,n+1}(\mathbb{P}^r, d) \rightarrow \overline{M}_{0,n}(\mathbb{P}^r, d)$  be the map that forgets mark  $i$ . Then

$$\begin{aligned}\pi_{i*}\Phi_i &= \tau \\ \pi_{i*}(\eta_i^2) &= \iota.\end{aligned}$$

Therefore, for  $a + b = rd + r + d - 3$  we have

$$\int \eta_1^2 \cdots \eta_a^2 \cdot \Phi_{a+1} \cdots \Phi_{a+b} \cap [\overline{M}_{0,a+b}(\mathbb{P}^r, d)] = \int \iota^a \tau^b \cap [\overline{M}_{0,0}(\mathbb{P}^r, d)],$$

by the projection formula.

**4.11 Example.** (Simple characteristic numbers of rational plane curves.) — For  $a + b = 3d - 1$ , let  $N_d(a, b)$  denote the number of rational plane curves passing through  $a$  points and tangent to  $b$  lines. By the corollary, it is the number

$$\int \eta_1^2 \cdots \eta_a^2 \cdot \Phi_{a+1} \cdots \Phi_{a+b} \cap [\overline{M}_{0,a+b}(\mathbb{P}^2, d)].$$

**4.12 Example.** — In  $\mathbb{P}^2$ , the condition of being tangent to a given line at a specified point is simply  $\Phi_i \eta_i$ . As in Ernström and Kennedy [2], let  $N_d(a, b, c)$  denote the number of rational curves passing through  $a$  points, tangent to  $b$  lines, and tangent to  $c$  lines at specified points.

By the corollary, the characteristic number  $N_d(a, b, c)$  is the top intersection

$$\int_{\overline{M}_{0,a+b+c}(\mathbb{P}^2, d)} \eta_1^2 \cdots \eta_a^2 \cdot \Phi_{a+1} \cdots \Phi_{a+b} \cdot \Phi_{a+b+1} \eta_{a+b+1} \cdot \Phi_{a+b+c} \eta_{a+b+c}$$

Computing these top products by recursion of twisted descendants reproduces the numbers of Ernström and Kennedy [2].

## Twisted cubics

Just to close the exposition with some *numbers*, I present three tables of characteristic numbers of twisted cubics in  $\mathbb{P}^3$ , most of which, to the best of my knowledge, could not have been computed with previous techniques. The discussion above immediately yields the computability of the characteristic numbers  $N_3(a, b, c, d, e)$  one can make out of the following five conditions: (a) incident to a line; (b) tangent to a plane; (c) passing through a point; (d) tangent to a plane at a specified line; and (e) tangent to a plane at a specified point. There are 249 such numbers; 7 of which are Gromov-Witten invariants; 42 others have  $d = e = 0$ . Most of these 49 numbers were computed by Schubert [16], and they were verified to modern standards of rigour by Kleiman, Strømme, and Xambó [11].

Although Schubert considers also the two compound conditions ( $d > 0$  or  $e > 0$ ), he only computes 6 out of the remaining 200 numbers, namely those with  $c = 4$  or  $c = 5$ , and he attributes them to Sturm:

$$\begin{aligned}
N_3(0, 0, 5, 1, 0) &= 2 \\
N_3(1, 0, 4, 0, 1) &= 3 \\
N_3(0, 1, 4, 0, 1) &= 6 \\
N_3(2, 0, 4, 1, 0) &= 17 \\
N_3(1, 1, 4, 1, 0) &= 34 \\
N_3(0, 2, 4, 1, 0) &= 68,
\end{aligned}$$

The techniques of the present note confirm these numbers; for example, the number  $N_3(1, 1, 4, 1, 0)$  of twisted cubics passing 4 points, incident to 1 line, tangent to 1 plane, and tangent to 1 plane at a given line is

$$\eta_1^3 \eta_2^3 \eta_3^3 \eta_4^3 \cdot \eta_5^2 \cdot \eta_6(\eta_6 + \overline{\psi}_6) \cdot \eta_7^2(\eta_7 + \overline{\psi}_7) = 34.$$

The number  $N_3(1, 0, 4, 0, 1) = 3$  was previously confirmed by Gathmann [6] using Gromov-Witten invariants of projective space blown up in a point. (His method yields, for any  $r \geq 2$ ,  $d \geq 2$ , the number of rational curves passing one specified point with tangent direction contained in a specified linear space, and incident to further linear spaces to get correct codimension.)

**4.13 Example.** — *Conditions w.r.t. planes and points:* the numbers  $N_3(0, b, c, 0, e)$  of twisted cubics

- tangent to  $e$  planes at specified points,
- passing through  $c$  points, and
- tangent to further  $b = 12 - 3e - 2c$  planes (at any point):

	$e = 0$	$e = 1$	$e = 2$	$e = 3$	$e = 4$
$c = 0$	56960	5552	816	64	4
$c = 1$	25344	3024	240	12	
$c = 2$	11968	888	40		
$c = 3$	3376	136	2		
$c = 4$	480	6			
$c = 5$	20				
$c = 6$	1				

$b = 12 - 3e - 2c$

The numbers of the first column are well-known through [16] and [11], and could also be computed using the algorithm of [13].

**4.14 Example.** — *Conditions w.r.t. planes and lines:* the numbers  $N_3(a, b, 0, d, 0)$  of twisted cubics

- tangent to  $d$  planes at specified lines,
- tangent to further  $b$  planes (at any point), and
- incident to  $a = 12 - 2d - b$  lines:

	$d = 0$	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$
$b = 0$	80160	38568	17472	7234	2630	805	217
$b = 1$	134400	60208	24720	8972	2780	730	
$b = 2$	209760	85216	30864	9616	2552	580	
$b = 3$	297280	107024	33472	8952	2064		
$b = 4$	375296	117312	31584	7344	1504		
$b = 5$	415360	112128	26272	5424			
$b = 6$	401920	94528	19680	3712			
$b = 7$	343360	71776	13664				
$b = 8$	264320	50464	9152				
$b = 9$	188256	34032					
$b = 10$	128160	22688					
$b = 11$	85440						
$b = 12$	56960						

$$a = 12 - 2d - b$$

Again, the first column is well-known through [16], [11] and [13].

**4.15 Example.** — *Combinations of the three tangency conditions:* the numbers  $N_3(0, b, 0, d, e)$  of twisted cubics

- tangent to  $e$  planes at specified points,
- tangent to  $d$  planes at specified lines, and
- tangent to further  $b = 12 - 3e - 2d$  planes (at any point):

	$e = 0$	$e = 1$	$e = 2$	$e = 3$	$e = 4$
$d = 0$	56960	5552	816	64	4
$d = 1$	22688	2320	240	16	
$d = 2$	9152	904	72		
$d = 3$	3712	328	20		
$d = 4$	1504	110			
$d = 5$	580				
$d = 6$	217				

$$b = 12 - 3e - 2d$$

(The first number in this table is well-known. . . )

## References

- [1] W. BRUNS and J. HERZOG, *Cohen-Macaulay rings*. Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1993.
- [2] L. ERNSTRÖM and G. KENNEDY, *Recursive formulas for the characteristic numbers of rational plane curves*. J. Alg. Geo. **7** (1998), 141–181. (Also available at alg-geom/9604019).
- [3] L. ERNSTRÖM and G. KENNEDY, *Contact cohomology of the projective plane*. Preprint, alg-geom/9703013.

- [4] W. FULTON, *Intersection theory*. Springer-Verlag, Berlin-Heidelberg, 1984.
- [5] W. FULTON and R. PANDHARIPANDE, *Notes on stable maps and quantum cohomology*. In *Algebraic Geometry, Santa Cruz 1995*, J. Kollár, R. Lazarsfeld and D. Morrison, eds., Proc. Symp. Pure. Math., **62** (II), 45–96. (Also available at alg-geom/9608011).
- [6] A. GATHMANN, *Gromov-Witten invariants of blow-ups*. Preprint, math.AG/9804043.
- [7] E. GETZLER, *Topological recursion relations in genus 2*. Preprint, math.AG/9801003.
- [8] E. GETZLER, *The Virasoro conjecture for Gromov-Witten invariants*. Preprint, math.AG/9812026.
- [9] T. GRABER and R. PANDHARIPANDE, in preparation.
- [10] S. KLEIMAN, *The transversality of a general translate*. Comp. Math. **28** (1974), 287–297.
- [11] S. KLEIMAN, S. A. STRØMME, and S. XAMBÓ, *Sketch of a verification of Schubert’s number 5819539783680 of twisted cubics*. In: *Space Curves, Proceedings, Rocca di Papa 1985*. LNM 1266, Springer-Verlag, New York.
- [12] M. KONTSEVICH and YU. I. MANIN, *Relations between the correlators of the topological sigma-model coupled to gravity*. Preprint, alg-geom/9708024.
- [13] R. PANDHARIPANDE, *Intersections of  $\mathbb{Q}$ -divisors on Kontsevich’s moduli space  $\overline{M}_{0,n}(\mathbb{P}^r, d)$  and enumerative geometry*. Preprint, alg-geom/9504004.
- [14] R. PANDHARIPANDE, *Rational curves on hypersurfaces (after A. Givental)*. Preprint, math.AG/9806133.
- [15] R. PANDHARIPANDE, Unpublished e-mail to Lars Ernström, May 2, 1997.
- [16] H. SCHUBERT, *Kalkül der abzählenden Geometrie*. Teubner, Leipzig, 1879. (Reprinted by Springer-Verlag, 1979).
- [17] E. WITTEN, *Two-dimensional gravity and intersection theory on moduli space*. Surveys in Diff. Geom. **1** (1991), 243–310.

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